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ON EQUIVALENCY OF VARIOUS EXPRESSIONS FOR SPEED OF WAVE PROPAGATION FOR COMPRESSIBLE LIQUID FLOWS WITH HEAT TRANSFER

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Abstract—It is demonstrated that for a compressible flow model with heat transfer, the introduction of a specific state equation to supplement the continuity, momentum and enthalpy equations, leads to a very specific form of an expression for a speed of wave propagation. Consequently, the numerous expressions obtained for various choices of state equations are not easily identifiable and, therefore, can not be evaluated directly in terms of measurable properties. By utilizing the various thermodynamic relationships, we have shown that these expressions are all equivalent and are identifiable as isentropic sonic velocity. As a corollary to this demonstration, we have also obtained expressions in terms of measurable properties for various thermodynamic-state variables occurring in the coefficients of the governing equations. These expressions are required if loss in accuracy owing to noise introduced in the direct numerical differentiation of the derivatives that these state-variables represent is to be avoided.

NOMENCLATURE

A, flow cross-sectional area;
a, speed wave propagation defined by equation (11);
C_p, $= (\partial h/\partial T)_P$, specific heat at constant pressure;
C_v, $= (\partial u/\partial T)_\rho$, specific heat at constant volume;
c, speed of wave propagation defined by equation (15);
g, acceleration due to gravity;
h, specific enthalpy;
P, pressure;
Q_w, $= q_w S_w/A$;
q_w, heat flux;
R_h, $= (\partial\rho/\partial h)_P$;
R_p, $= (\partial\rho/\partial P)_h$;
S_w, heated or wetted perimeter;
s, specific entropy;
T, temperature of a liquid coolant;
t, time;
u, specific internal energy;
v, fluid velocity;
z, coordinate in the vertical direction.

Greek symbols

α_p, $-(\partial\rho/\partial T)_P/\rho$, coefficient of thermal expansion;
β_T, $(\partial\rho/\partial P)_T/\rho$, isothermal coefficient of bulk compressibility;
β_s, $(\partial\rho/\partial P)_s/\rho$, adiabatic coefficient of bulk compressibility;
γ_v, $(\partial P/\partial T)_\rho$, thermal pressure coefficient;
Λ_T, $(\partial h/\partial P)_T$;
λ_p, $1/R_h$;
λ_ρ, $(\partial h/\partial P)_\rho$;
ρ, fluid density;
τ_w, wall shear stress;
Φ_w, $v\tau_w S_w/A$;
Ω, speed of wave propagation as defined by equation (22).

INTRODUCTION

THE COMPRESSIBLE liquid flow with heat transfer occurs in numerous nuclear reactor applications. For example, loss of flow resulting from pipe rupture both in the case of boiling water and pressurized water reactors, and also in the case of liquid metal cooled fast breeder reactors (LMFBRs) requires the modeling of compressible liquid flow with heat transfer. The dynamics of the coolant, subsequently to the release of molten fuel in the coolant channels during a power transient in an LMFBR, are generally analyzed in terms of compressible coolant flow with heat transfer. The governing equations are solved either by the use of finite-difference methods or by the method of characteristics. In both schemes, one needs to determine the speed of wave propagation as a function of tabulated properties of the liquids. In the explicit form of finite-difference methods (such as Lax method, two step Lax–Wendroff difference method, donor-cell type different method), one needs speed of wave propagation to determine the size of time steps that satisfies stability criterion [1]. In the case of method of characteristics, the speed of wave propagation is required to determine the slopes of the characteristics. In this latter application, the form of the expression for the speed of wave propagation varies with the exact form of governing equations chosen for solution. Very frequently, these expressions have very diverse form and do not permit direct numerical evaluation in terms of the properties tabulated in the standard tables. In addition, the algebraic manipulation involved to obtain the governing equations in a form suitable for application of a given numerical method (such as the method of characteristics) leads to very complex forms of coefficients of the various terms of the resultant governing equations. These coefficients, generally, are both function of flow parameters and of the physical properties of the coolant. The purpose of the present note is to demonstrate various expressions arrived at in most common formulations of the governing equations for compressible liquid flow with heat transfer are equivalent and are identifiable with very familiar expression, namely $a^2 = (\partial P/\partial \rho)_s = 1/\rho\beta_s$ (see Nomenclature for definition of symbols) for the speed of sound defined for the isentropic process of state change and fluid flow. As a corollary to this demonstration, we will show how the various coefficients which depend only on the properties, can be expressed in a form suitable for direct numerical evaluation by the use of tabulated or measured properties.

FORMULATION

The most common form of the basic governing equations, which are utilized to describe flow through a coolant channel of constant cross-sectional area during a transient such as initiated by loss of flow due to pipe rupture, is [2–4]:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial z}(\rho v) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial z}(\rho v^2) = -\frac{\partial P}{\partial z}-\rho g-\frac{S_w \tau_w}{A}, \tag{2}$$

$$\frac{\partial}{\partial t}(\rho h)+\frac{\partial}{\partial z}(\rho v h) = \frac{q_w S_w}{A}+\frac{v\tau_w S_w}{A}+\frac{\partial P}{\partial t}+v\frac{\partial P}{\partial z}, \tag{3}$$

$$\rho = \rho(h, P), \tag{4a}$$

or

$$\rho = \rho(T, P), \tag{4b}$$

$$h = h(\rho, P), \tag{4c}$$

or

$$h = h(T, P). \tag{4d}$$

If we utilize the first equation of state (4a), we can make equations (1)–(3) explicit in variables v, P, h. The use of third state equation (4c) will render equations (1)–(3) explicit in ρ, v and P. Depending upon the choice of a specific form of state equation as we will illustrate below, we arrive at different expressions for the speed of wave propagation.

Explicit in variable v, P, h

The use of first state equation (4a) in equation (1) gives

$$R_h\left(\frac{\partial h}{\partial t}+v\frac{\partial h}{\partial z}\right)+R_p\left(\frac{\partial P}{\partial t}+v\frac{\partial P}{\partial z}\right)+\rho\frac{\partial v}{\partial z} = 0, \tag{5}$$

where

$$R_h = \left(\frac{\partial \rho}{\partial h}\right)_P \quad R_p = \left(\frac{\partial \rho}{\partial P}\right)_h. \tag{6}$$

The use of equation (1) into equations (2) and (3) yields,

$$\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial z}-\rho g-\frac{S_w \tau_w}{A}, \tag{7}$$

$$\rho\left(\frac{\partial h}{\partial t}+v\frac{\partial h}{\partial z}\right) = Q_w+\Phi_w+\left(\frac{\partial P}{\partial t}+v\frac{\partial P}{\partial z}\right), \tag{8a}$$

where

$$Q_w = \frac{q_w S_w}{A}, \quad \Phi_w = \frac{v\tau_w S_w}{A}. \tag{8b}$$

Combining equations (5) and (8) first to eliminate h between them and second to eliminate P between them:

$$\frac{\partial P}{\partial t}+v\frac{\partial P}{\partial z}+\rho a^2\frac{\partial v}{\partial z}+\frac{R_h a^2}{\rho}(Q_w+\Phi_w) = 0, \tag{9}$$

$$\frac{\partial h}{\partial t}+v\frac{\partial h}{\partial z}+a^2\frac{\partial v}{\partial z} = \frac{R_p a^2}{\rho}(Q_w+\Phi_w), \tag{10}$$

where

$$a = \frac{1}{(R_p+R_h/\rho)^{1/2}}. \tag{11}$$

We will show, subsequently, that the above expression defines the isentropic sonic velocity corresponding to the system of equations (7), (9) and (10) which are explicit in v, P and h. We may note, here, that properties R_p and R_h as defined by equation (6) are not directly measurable properties. It is clear that the numerical evaluation of these properties is required not only for the determination of quantity a but also for determining coefficients in the governing equations (9) and (10).

Explicit in variables ρ, v, P.

From the state equation (4c), we have

$$\mathrm{d}h = \lambda_p \mathrm{d}\rho+\lambda_\rho \mathrm{d}P. \tag{12}$$

The use of equation (12) into equation (8) yields

$$\rho\lambda_p\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial z}\right)+(\rho\lambda_\rho-1)\left(\frac{\partial P}{\partial t}+v\frac{\partial P}{\partial z}\right) = Q_w+\Phi_w. \tag{13}$$

Eliminating the terms involving ρ in equation (13) by using equation (1), we obtain

$$\left(\frac{\partial P}{\partial t}+v\frac{\partial P}{\partial z}\right)+\rho c^2\frac{\partial v}{\partial z} = \frac{1}{\rho\lambda_\rho-1}(Q_w+\Phi_w), \tag{14}$$

where

$$c = \left(\frac{-\rho\lambda_p}{\rho\lambda_\rho-1}\right)^{1/2}. \tag{15}$$

The above expression also defines isentropic sonic velocity, although the form of the expression is not a very familiar one. In the subsequent analysis, we demonstrate that the above is indeed isentropic sonic velocity. Once again, properties λ_p and λ_ρ are not directly measurable quantities, therefore, unless we can demonstrate that expression (15) indeed represents an isentropic sonic velocity, we will not be able to calculate satisfactorily the numerical values for the quantity c. Thus, if one wants to utilize the system of equations (1), (7) and (14) which are explicit in ρ, v and P, one must numerically evaluate the quantity c as defined by expression (15).

Explicit in variables v, P, T

From state equation (4b), we can write

$$d\rho = -\rho\alpha_p dT + \rho\beta_T dP. \tag{16}$$

The use of equation (16) into equation (1) yields

$$-\rho\alpha_p\left(\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial z}\right)+\rho\beta_T\left(\frac{\partial P}{\partial t}+v\frac{\partial P}{\partial z}\right)+\rho\frac{\partial v}{\partial z}=0. \tag{17}$$

From state equation (4d), we have

$$dh = C_p dT + \Lambda_T dP. \tag{18}$$

The use of equation (18) into equation (8) gives

$$\rho C_p\left(\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial z}\right)+(\rho\Lambda_T-1)\left(\frac{\partial P}{\partial t}+v\frac{\partial P}{\partial z}\right)=Q_w+\Phi_w. \tag{19}$$

The combination of equations (17) and (19) gives

$$\frac{\partial P}{\partial t}+v\frac{\partial P}{\partial z}+\rho\Omega^2\frac{\partial v}{\partial z}=\frac{\alpha_p\Omega^2}{C_p}(Q_w+\Phi_w), \tag{20}$$

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial z}-\frac{\rho\Lambda_T-1}{C_p}\Omega^2\frac{\partial v}{\partial z}=\frac{\beta_T\Omega^2}{C_p}(Q_w+\Phi_w), \tag{21}$$

where

$$\Omega=\frac{1}{\left(\rho\beta_T+\frac{\rho\Lambda_T-1}{C_p}\alpha_p\right)^{1/2}}. \tag{22}$$

Equations (20), (21) together with equation (7) represents a system of equations that are explicit in P, T and v. This system has necessitated the introduction of parameter Ω as defined by equation (22). Although not recognizable as it is defined in the form (22), but it will be shown subsequently that Ω is the isentropic sonic velocity. We, also, further recognize in view of the third term of equation (21) the need for numerical evaluation of quantity Λ_T.

EVALUATION OF VARIOUS PARAMETERS IN TERMS OF MEASURABLE PROPERTIES

Various thermodynamic-state variables such as a, c, Ω, R_p, λ_p, λ_ρ, Λ_T, and R_h as introduced previously must be expressed in terms of measurable or derived properties for their evaluation.

Parameters R_h *and* λ_p

From the definition of the parameter R_h as given by equation (6) and from the definitions of α_p and C_p, we can write

$$R_h=\left(\frac{\partial\rho}{\partial T}\right)_P\left(\frac{\partial T}{\partial h}\right)_P=-\frac{\rho\alpha_p}{C_p}. \tag{23a}$$

From equation (23a), we obtain

$$\lambda_p=-\frac{C_p}{\rho\alpha_p}. \tag{23b}$$

Equation (23) will permit us to determine R_h and λ_p since ρ, C_p and α_p are either tabulated in properties tables (see for example Padilla [5]) or can be derived from other measurable properties [5, 6].

Parameters R_p *and* λ_ρ

From equations (4a) and (12), we can obtain

$$R_p=\left(\frac{\partial\rho}{\partial P}\right)_h=-\frac{\left(\frac{\partial h}{\partial P}\right)_\rho}{\left(\frac{\partial h}{\partial\rho}\right)_P}=-R_h\left(\frac{\partial h}{\partial P}\right)_\rho. \tag{24}$$

The use of thermodynamic relationship

$$h=u+P/\rho \tag{25}$$

yields

$$\left(\frac{\partial h}{\partial P}\right)_\rho=\frac{C_v}{\gamma_v}+\frac{1}{\rho}. \tag{26}$$

From the state equations (4), one can readily derive.

$$\gamma_v=\left(\frac{\partial P}{\partial T}\right)_\rho=-\frac{\left(\frac{\partial\rho}{\partial T}\right)_P}{\left(\frac{\partial\rho}{\partial P}\right)_T}=\frac{\alpha_P}{\beta_T}. \tag{27}$$

Substituting for γ_v from equation (27) and for $C_v = C_p\beta_s/\beta_T$ (see [6] for the derivation of this relationship) into equation (26), we have

$$\left(\frac{\partial h}{\partial P}\right)_\rho=C_p\frac{\beta_s}{\alpha_p}+\frac{1}{\rho}. \tag{28}$$

The use of the following relationship whose derivation can be found in [6] and [7]

$$\beta_T-\beta_s=\frac{T\alpha_p^2}{\rho C_p} \tag{29}$$

into equation (28) yields finally

$$\lambda_\rho=\left(\frac{\partial h}{\partial P}\right)_\rho=\frac{T\alpha_p\beta_s}{\rho(\beta_T-\beta_s)}+\frac{1}{\rho}. \tag{30}$$

Substituting equations (23), (30) and (29) into equation (24) gives

$$R_p=\rho\beta_s+\frac{\alpha_p}{C_p}. \tag{31}$$

The relationships (30) and (31) will enable us to evaluate λ_ρ and R_p, respectively, in terms of either directly measurable properties or properties that can be derived from other measured properties (see for example [5]). For example, β_s is directly measurable or can be determined from the measurements of sonic velocity [5, 6].

The use of equation (31) and (23a) into equation (11) gives

$$a = \frac{1}{\left(R_p + \frac{R_h}{\rho}\right)^{1/2}} = \frac{1}{(\rho\beta_s)^{1/2}} = \left(\frac{\partial P}{\partial \rho}\right)_s^{1/2}. \quad (32)$$

The substitution of equations (23b), (30) and (29) into equation (15) yields

$$c = \left(\frac{-\rho\lambda_p}{\rho\lambda_\rho - 1}\right)^{1/2} = \frac{1}{(\rho\beta_s)^{1/2}} = \left(\frac{\partial P}{\partial \rho}\right)_s^{1/2}. \quad (33)$$

Clearly, expressions (11) and (15) have become identifiable namely, each of them defines isentropic sonic velocity.

Substituting the following relation for Λ_T whose derivation can be found in [6],

$$\Lambda_T = \left(\frac{\partial h}{\partial P}\right)_T = \frac{1}{\rho}(1 - T\alpha_p), \quad (34)$$

and equation (29) into equation (22) gives

$$\Omega = \frac{1}{(\rho\beta_s)^{1/2}} = \left(\frac{\partial P}{\partial \rho}\right)_s^{1/2} \quad (35)$$

The expression (22), for Ω, thus, becomes recognizable.

CONCLUSIONS

By utilizing various thermodynamic relations, we have been able to demonstrate that the various expressions for the speed of wave propagation introduced due to specific choices of a state equation, are identifiable with isentropic sonic velocity and, therefore, can be evaluated in terms of measurable properties. We have also derived expressions for various thermodynamic-state variables occurring in the coefficients of the governing equation, in terms of measurable and/or derived properties. The need for such expressions can hardly be overemphasized in view of the fact that direct numerical calculations of the derivatives that these state variables represent will introduce noise and, therefore, their accuracy will be suspect.

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SUR L'EQUIVALENCE DE PLUSIEURS EXPRESSIONS DE LA VITESSE DE PROPAGATION D'UNE ONDE POUR DES ECOULEMENTS DE LIQUIDES COMPRESSIBL ES AVEC TRANSFERT DE CHALEUR

Résumé—On montre que pour un modèle d'écoulement compressible avec transfert de chaleur, l'introduction d'une équation d'état spécifique en supplément des équations de continuité, de quantit de mouvement et d'enthalpie, conduit à une forme très spécifique d'expression pour la vitesse de propagation d'une onde. En conséquence, les nombreuses expressions obtenues pour différents choix d'équations d'état ne sont pas aisément identifiables et ne peuvent être évaluées directement en fonction des propriétés mesurables. En utilisant les relations de la thermodynamique, il est montré que ces expressions sont toutes équivalentes et identifiables à la vitesse du son isentropique. Comme corrolaire à cette démonstration, des expressions sont obtenues en termes de propriétés mesurables pour plusieurs variables thermodynamiques d'état intervenant dans les coefficients des équations du problème. Ces expressions sont nécessaires si l'on veut éviter la perte de précision introduite dans la différentiation numérique directe de grandeurs dérivées que les variables d'état représentent.

ZUR GLEICHWERTIGKEIT VERSCHIEDENER AUSDRÜCKE FÜR DIE GESCHWINDIGKEIT DER WELLENAUSBREITUNG BEI KIMPRESSIBLEN FLÜSSIGKEITSSTRÖMUNGEN MIT WÄRMEÜBERGANG

Zusammenfassung—Es wird gezeigt, daß die Einführung einer spezifischen Zustandsgleichung zur Ergänzung der Kontinuitäts-, Impuls- und Enthalpiegleichungen bei dem Modell einer kompressiblen Strömung mit Wärmeübergang zu einer sehr spezifischen Gleichungsform für die Wellenausbreitungsgeschwindigkeit führt. Folglich lassen sich die zahlreichen Ausdrücke, die man bei Verwendung der unterschiedlichen Zustandsgleichungen erhält, nicht einfach als gleich erkennen und können deshalb auch nicht direkt nach meßbaren Größen aufgelöst werden. Durch die Anwendung der verschiedenen thermodynamischen Beziehungen haben wir gezeigt, daß alle diese Ausdrücke gleichwertig sind und als isentrope Schallgeschwindigkeit gedeutet werden können. Als Folge dieser Darstellung erhielten wir auch Ausdrücke aus meßbaren Eigenschaften für verschiedene thermodynamische Zustandsvariablen, die in den Koeffizienten der maßgebenden Gleichungen auftreten. Diese Ausdrücke werden benötigt, wenn Genauigkeitseinbußen vermieden werden sollen, die auf das Auftreten von Instabilitäten bei der direkten numerischen Differentiation der Ableitungen, die diese Zustandsvariablen darstallen, zurückzuführen sind.

ОБ ЭКВИВАЛЕНТНОМ ХАРАКТЕРЕ ВЫРАЖЕНИЙ ДЛЯ СКОРОСТИ РАСПРОСТРАНЕНИЯ ВОЛНЫ В ПОТОКЕ СЖИМАЕМОЙ ЖИДКОСТИ ПРИ НАЛИЧИИ ТЕПЛООБМЕНА

Аннотация — Показано, что при описании течения сжимаемой жидкости с теплообменом добавление конкретного уравнения состояния к уравнениям сохранения массы, импульса и энергии приводит к очень специфической форме выражения для скорости распространения волны. Вследствие этого многочисленные выражения, полученные при использовании различных уравнений состояния, оказываются трудно идентифицируемыми и непригодными для проведения расчётов в терминах непосредственно измеряемых величин. Используя различные термодинамические соотношения, авторы показали, что все эти выражения являются эквивалентными и определяют из энтропическую скорость звука. Для подтверждения данного вывода получены выражения в терминах непосредственно измеряемых величин для различных термодинамических параметров, входящих в коэффициенты исходных уравнений. Эти выражения требуются в тех случаях, когда необходимо избежать потери точности из-за возникающих погрешностей при прямом численном дифференцировании производных, представленных указанными параметрами состояния.